AP Calculus BC

WS 76 - Convergence of Inf. Series

1) $\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} = \sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n$

$S_0 = 1 \quad S_1 = 1 - \frac{1}{4} = \frac{3}{4} \quad S_2 = \frac{3}{4} + \frac{1}{16} = \frac{13}{16}$

$S_3 = \frac{13}{16} - \frac{1}{64} = \frac{51}{64}$

$S = \frac{1}{1+\frac{1}{4}} = \frac{1}{5/4} = \frac{4}{5}$

2) $\sum_{n=1}^{\infty} \frac{7}{4^n} = \sum_{n=1}^{\infty} 7 \cdot \left(\frac{1}{4}\right)^n$

$S_1 = \frac{7}{4} \quad S_2 = \frac{7}{4} + \frac{7}{16} = \frac{35}{16} \quad S_3 = \frac{35}{16} + \frac{7}{64} = \frac{147}{64}$

$S_4 = \frac{147}{64} + \frac{7}{256} = \frac{595}{256}$

$S = \frac{7/4}{1 - 1/4} = \frac{7/4}{3/4} = \frac{7}{3}$

3) $\sum_{n=0}^{\infty} (-1)^n \frac{5}{4^n} = \sum_{n=0}^{\infty} 5\left(-\frac{1}{4}\right)^n$

$S_0 = 5 \quad S_1 = \frac{15}{4} \quad S_2 = \frac{65}{16} \quad S_3 = \frac{255}{64}$

$S = \frac{5}{1 + 1/4} = \frac{5}{5/4} = 4$

4) $\sum_{n=0}^{\infty} \left(\frac{2^{n+1}}{5^n}\right) = \sum_{n=0}^{\infty} 2 \cdot \left(\frac{2}{5}\right)^n$

$S_0 = 2 \quad S_1 = \frac{14}{5} \quad S_2 = \frac{78}{25} \quad S_3 = \frac{406}{125}$

$S = \frac{2}{1 - 2/5} = \frac{2}{3/5} = \frac{10}{3}$

5) $\sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n = S$

$r = \frac{1}{\sqrt{2}} < 1$

S converges by GST

$S = \frac{1}{1 - 1/\sqrt{2}} = \frac{1}{\sqrt{2} - 1/\sqrt{2}} = \boxed{\frac{\sqrt{2}}{\sqrt{2}-1}}$

6) $\sum_{n=0}^{\infty} \cos(n\pi) = S$

$\lim_{n\to\infty} \cos(n\pi)$ diverges

S diverges by n^{th} term test

7) $\sum_{n=1}^{\infty} \ln \frac{1}{n} = S$

$\lim_{n\to\infty} \left(\ln \frac{1}{n}\right)$ diverges

S diverges by n^{th} term test

8) $\sum_{n=0}^{\infty} \frac{n!}{1000^n} = S$

$\lim_{n\to\infty} \frac{n!}{1000^n} = \infty$

S diverges by n^{th} term test

9) $\sum_{n=0}^{\infty} \left(\frac{e}{\pi}\right)^n = S$

$r = \frac{e}{\pi} < 1$

S converges by GST

$S = \frac{1}{1 - e/\pi} = \frac{\pi}{\pi - e}$

10) $\sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n = S$

$r = \frac{1}{10} < 1$

S converges by GST

$S = \frac{1/10}{1 - 1/10} = \frac{1/10}{9/10} = \frac{1}{9}$

11) $\sum_{n=1}^{\infty} \frac{n}{n+1} = S$

$\lim_{n\to\infty} \frac{n}{n+1} = 1$

S diverges by n^{th} term test

12) $\sum_{n=1}^{\infty} \frac{5}{n+1} = S$

$\int_1^{\infty} \frac{5}{x+1}\, dx = \lim_{b\to\infty} \int_1^b \frac{5}{x+1}$

$5 \lim_{b\to\infty} \left[\ln(x+1)\right]_1^b = 5 \lim_{b\to\infty} \left[\ln(b+1) - \ln 2\right]$

diverges

S diverges by Integral Test

13) $\sum_{n=1}^{\infty} \frac{3}{n^{1/2}} = 3 \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} = S$

$p = 1/2 < 1$, S diverges by p-series test

14) $\sum_{n=1}^{\infty} \frac{-2}{n\sqrt{n}} = -2 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} = S$

$p = 3/2 > 1$, S converges by p-series test

15) $\sum_{n=1}^{\infty} \left(-\frac{1}{8^n}\right) = \sum_{n=1}^{\infty} (-1)\left(\frac{1}{8}\right)^n = S$

$r = \frac{1}{8} < 1$, S converges by GST

$S = \frac{-1/8}{1-1/8} = \frac{-1/8}{7/8} = -\frac{1}{7}$

16) $\sum_{n=2}^{\infty} \frac{\ln n}{n} = S$

$\int_2^{\infty} \frac{1}{x} \cdot \ln x\, dx = \lim_{b\to\infty} \int_2^b \frac{1}{x} \ln x\, dx$

$\lim_{b\to\infty} \left[\frac{1}{2}(\ln x)^2\right]_2^b$

$\lim_{b\to\infty} \left[\frac{1}{2}(\ln b)^2 - \frac{1}{2}(\ln 2)^2\right] = \infty$

S diverges by Integral Test

17) $\sum_{n=1}^{\infty} \frac{2^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = S$

$r = 2/3 < 1$, S converges by GST

$S = \frac{2/3}{1-2/3} = \frac{2/3}{1/3} = 2$

18) $\sum_{n=0}^{\infty} \frac{-2}{n+1} = S$

$-2\int_0^{\infty} \frac{1}{x+1}\, dx = -2\left[\lim_{b\to\infty} \int_0^b \frac{1}{x+1}\, dx\right]$

$-2\left[\lim_{b\to\infty} \left[\ln(x+1)\right]\Big|_0^b\right]$

$-2\left[\lim_{b\to\infty} (\ln b+1) - 0\right] = \infty$

S diverger by Integral Test

19) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n = S$

$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

S diverges by n^{th} term test

20) $\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}} = S$

$\int_1^{\infty} \frac{e^x}{1+(e^x)^2}\, dx$

$\lim_{b\to\infty} \int_1^b \frac{e^x}{1+(e^x)^2}\, dx$

$\to \lim_{b\to\infty} \left[\arctan e^x\right]_1^b$

$\lim_{b\to\infty} \left[\arctan e^b - \arctan e\right]$

$\frac{\pi}{2} - \arctan e$

S converges by Integral Test