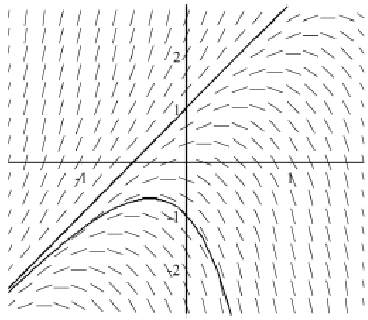
2001 BC5 (**5.61**)

(a) $\int_1^{\infty} -3x f(x)\,dx$ $= \int_1^{\infty} f'(x)\,dx = \lim_{b\to\infty} \int_1^{b} f'(x)\,dx = \lim_{b\to\infty} f(x)\Big\vert_1^b$ $= \lim_{b\to\infty} f(b) - f(1) = 0 - 4 = -4$	2 : { 1 : use of FTC 1 : answer from limiting process
(b) $f(1.5) \approx f(1) + f'(1)(0.5)$ $= 4 - 3(1)(4)(0.5) = -2$ $f(2) \approx -2 + f'(1.5)(0.5)$ $\approx -2 - 3(1.5)(-2)(0.5) = 2.5$	2 : { 1 : Euler's method equations or equivalent table 1 : Euler approximation to $f(2)$ (not eligible without first point)
(c) $\frac{1}{y}\,dy = -3x\,dx$ $\ln y = -\frac{3}{2}x^2 + k$ $y = Ce^{-3/2\,x^2}$ $4 = Ce^{-3/2}$; $C = 4e^{3/2}$ $y = 4e^{3/2} e^{-3/2\,x^2}$	5 : { 1 : separates variables 1 : antiderivatives 1 : constant of integration 1 : uses initial condition $f(1) = 4$ 1 : solves for y Note: max 2/5 [1-1-0-0-0] if no constant of integration Note: 0/5 if no separation of variables

2002 BC5 (**4.00**)

(a)



2 $\begin{cases} 1: \text{ solution curve through } (0,1) \\ 1: \text{ solution curve through } (0,-1) \end{cases}$

Curves must go through the indicated points, follow the given slope lines, and extend to the boundary of the slope field.

(b) $f(0.1) \approx f(0) + f'(0)(0.1)$

$= 1 + (2 - 0)(0.1) = 1.2$

$f(0.2) \approx f(0.1) + f'(0.1)(0.1)$

$\approx 1.2 + (2.4 - 0.4)(0.1) = 1.4$

2 $\begin{cases} 1: \text{ Euler's method equations or equivalent table applied to (at least) two iterations} \\ 1: \text{ Euler approximation to } f(0.2) \text{ (not eligible without first point)} \end{cases}$

(c) Substitute $y = 2x + b$ in the DE:

$2 = 2(2x + b) - 4x = 2b$, so $b = 1$

OR

Guess $b = 1$, $y = 2x + 1$

Verify: $2y - 4x = (4x + 2) - 4x = 2 = \dfrac{dy}{dx}$.

2 $\begin{cases} 1: \text{ uses } \dfrac{d}{dx}(2x + b) = 2 \text{ in DE} \\ 1: \ b = 1 \end{cases}$

(d) g has local maximum at $(0,0)$.

$g'(0) = \left.\dfrac{dy}{dx}\right|_{(0,0)} = 2(0) - 4(0) = 0$, and

$g''(x) = \dfrac{d^2y}{dx^2} = 2\dfrac{dy}{dx} - 4$, so

$g''(0) = 2\,g'(0) - 4 = -4 < 0$.

3 $\begin{cases} 1: \ g'(0) = 0 \\ 1: \text{ shows } g''(0) = -4 \\ 1: \text{ conclusion} \end{cases}$

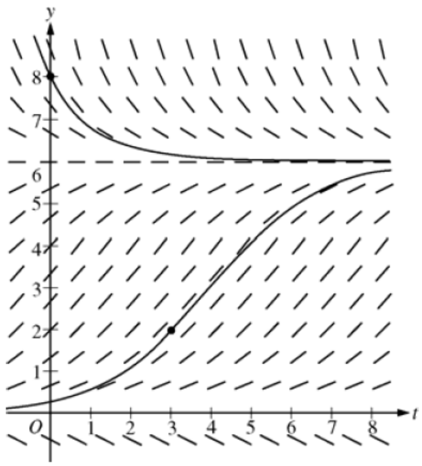
2004 BC5 (**3.23**)

(a) For this logistic differential equation, the carrying capacity is 12. If $P(0) = 3$, $\lim_{t\to\infty} P(t) = 12$. If $P(0) = 20$, $\lim_{t\to\infty} P(t) = 12$.	$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{answer} \end{cases}$
(b) The population is growing the fastest when P is half the carrying capacity. Therefore, P is growing the fastest when $P = 6$.	1 : answer
(c) $\frac{1}{Y}dY = \frac{1}{5}\left(1 - \frac{t}{12}\right)dt = \left(\frac{1}{5} - \frac{t}{60}\right)dt$ $\ln\lvert Y\rvert = \frac{t}{5} - \frac{t^2}{120} + C$ $Y(t) = Ke^{\frac{t}{5} - \frac{t^2}{120}}$ $K = 3$ $Y(t) = 3e^{\frac{t}{5} - \frac{t^2}{120}}$	$5 : \begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } Y \\ \quad 0/1 \text{ if } Y \text{ is not exponential} \end{cases}$ Note: max $2/5$ [1-1-0-0-0] if no constant of integration Note: $0/5$ if no separation of variables
(d) $\lim_{t\to\infty} Y(t) = 0$	1 : answer $0/1$ if Y is not exponential

2006 BC5 (**4.33**)

(a) $\left.\frac{dy}{dx}\right\rvert_{(-1,\,-4)} = 6$ $\frac{d^2y}{dx^2} = 10x + 6(y-2)^{-2}\frac{dy}{dx}$ $\left.\frac{d^2y}{dx^2}\right\rvert_{(-1,\,-4)} = -10 + 6\frac{1}{(-6)^2}6 = -9$	$3 : \begin{cases} 1 : \left.\frac{dy}{dx}\right\rvert_{(-1,\,-4)} \\ 1 : \frac{d^2y}{dx^2} \\ 1 : \left.\frac{d^2y}{dx^2}\right\rvert_{(-1,\,-4)} \end{cases}$
(b) The x-axis will be tangent to the graph of f if $\left.\frac{dy}{dx}\right\rvert_{(k,\,0)} = 0$. The x-axis will never be tangent to the graph of f because $\left.\frac{dy}{dx}\right\rvert_{(k,\,0)} = 5k^2 + 3 > 0$ for all k.	$2 : \begin{cases} 1 : \frac{dy}{dx} = 0 \text{ and } y = 0 \\ 1 : \text{answer and explanation} \end{cases}$
(c) $P(x) = -4 + 6(x+1) - \frac{9}{2}(x+1)^2$	$2 : \begin{cases} 1 : \text{quadratic and centered at } x = -1 \\ 1 : \text{coefficients} \end{cases}$
(d) $f(-1) = -4$ $f\left(-\frac{1}{2}\right) \approx -4 + \frac{1}{2}(6) = -1$ $f(0) \approx -1 + \frac{1}{2}\left(\frac{5}{4} + 2\right) = \frac{5}{8}$	$2 : \begin{cases} 1 : \text{Euler's method with 2 steps} \\ 1 : \text{Euler's approximation to } f(0) \end{cases}$

2008 BC6 (**3.84**)

(a) 	$2: \begin{cases} 1\text{: solution curve through } (0,8) \\ 1\text{: solution curve through } (3,2) \end{cases}$
(b) $f\left(\frac{1}{2}\right) \approx 8 + (-2)\left(\frac{1}{2}\right) = 7$ $f(1) \approx 7 + \left(-\frac{7}{8}\right)\left(\frac{1}{2}\right) = \frac{105}{16}$	$2: \begin{cases} 1 : \text{Euler's method with two steps} \\ 1 : \text{approximation of } f(1) \end{cases}$
(c) $\frac{d^2y}{dt^2} = \frac{1}{8}\frac{dy}{dt}(6-y) + \frac{y}{8}\left(-\frac{dy}{dt}\right)$ $f(0) = 8;\; f'(0) = \left.\frac{dy}{dt}\right\rvert_{t=0} = \frac{8}{8}(6-8) = -2;$ and $f''(0) = \left.\frac{d^2y}{dt^2}\right\rvert_{t=0} = \frac{1}{8}(-2)(-2) + \frac{8}{8}(2) = \frac{5}{2}$ The second-degree Taylor polynomial for f about $t = 0$ is $P_2(t) = 8 - 2t + \frac{5}{4}t^2$. $f(1) \approx P_2(1) = \frac{29}{4}$	$4: \begin{cases} 2 : \frac{d^2y}{dt^2} \\ 1 : \text{second-degree Taylor polynomial} \\ 1 : \text{approximation of } f(1) \end{cases}$
(d) The range of f for $t \ge 0$ is $6 < y \le 8$.	1 : answer

2009 BC4 (**4.7**)

(a) $f\left(-\frac{1}{2}\right) \approx f(-1) + \left(\left.\frac{dy}{dx}\right|_{(-1,\,2)}\right) \cdot \Delta x$

$= 2 + 4 \cdot \frac{1}{2} = 4$

$f(0) \approx f\left(-\frac{1}{2}\right) + \left(\left.\frac{dy}{dx}\right|_{\left(-\frac{1}{2},\,4\right)}\right) \cdot \Delta x$

$\approx 4 + \frac{1}{2} \cdot \frac{1}{2} = \frac{17}{4}$

2 : { 1 : Euler's method with two steps; 1 : answer }

(b) $P_2(x) = 2 + 4(x+1) - 6(x+1)^2$

1 : answer

(c) $\frac{dy}{dx} = x^2(6 - y)$

$\int \frac{1}{6-y}\, dy = \int x^2\, dx$

$-\ln|6-y| = \frac{1}{3}x^3 + C$

$-\ln 4 = -\frac{1}{3} + C$

$C = \frac{1}{3} - \ln 4$

$\ln|6-y| = -\frac{1}{3}x^3 - \left(\frac{1}{3} - \ln 4\right)$

$|6-y| = 4e^{-\frac{1}{3}\left(x^3+1\right)}$

$y = 6 - 4e^{-\frac{1}{3}\left(x^3+1\right)}$

6 : { 1 : separation of variables; 2 : antiderivatives; 1 : constant of integration; 1 : uses initial condition; 1 : solves for y }

Note: max $3/6$ [1-2-0-0-0] if no constant of integration

Note: $0/6$ if no separation of variables

2010 BC5 (**4.54**)

(a) $f\left(\frac{1}{2}\right) \approx f(1) + \left(\left.\frac{dy}{dx}\right\|_{(1,\,0)}\right) \cdot \Delta x$ $= 0 + 1 \cdot \left(-\frac{1}{2}\right) = -\frac{1}{2}$ $f(0) \approx f\left(\frac{1}{2}\right) + \left(\left.\frac{dy}{dx}\right\|_{\left(\frac{1}{2},\,-\frac{1}{2}\right)}\right) \cdot \Delta x$ $\approx -\frac{1}{2} + \frac{3}{2} \cdot \left(-\frac{1}{2}\right) = -\frac{5}{4}$	2 : { 1 : Euler's method with two steps 1 : answer
(b) Since f is differentiable at $x = 1$, f is continuous at $x = 1$. So, $\lim_{x \to 1} f(x) = 0 = \lim_{x \to 1}\left(x^3 - 1\right)$ and we may apply L'Hospital's Rule. $\lim_{x \to 1} \frac{f(x)}{x^3 - 1} = \lim_{x \to 1} \frac{f'(x)}{3x^2} = \frac{\lim_{x \to 1} f'(x)}{\lim_{x \to 1} 3x^2} = \frac{1}{3}$	2 : { 1 : use of L'Hospital's Rule 1 : answer
(c) $\frac{dy}{dx} = 1 - y$ $\int \frac{1}{1 - y}\, dy = \int 1\, dx$ $-\ln\|1 - y\| = x + C$ $-\ln 1 = 1 + C \Rightarrow C = -1$ $\ln\|1 - y\| = 1 - x$ $\|1 - y\| = e^{1-x}$ $f(x) = 1 - e^{1-x}$	5 : { 1 : separation of variables 1 : antiderivatives 1 : constant of integration 1 : uses initial condition 1 : solves for y Note: max $2/5$ [1-1-0-0-0] if no constant of integration Note: $0/5$ if no separation of variables

2013 BC5 (**5.59**)

(a) $\lim_{x\to 0}(f(x)+1) = -1+1 = 0$ and $\lim_{x\to 0}\sin x = 0$

Using L'Hospital's Rule,

$$\lim_{x\to 0}\frac{f(x)+1}{\sin x} = \lim_{x\to 0}\frac{f'(x)}{\cos x} = \frac{f'(0)}{\cos 0} = \frac{(-1)^2\cdot 2}{1} = 2$$

$2 : \begin{cases} 1 : \text{L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$

(b) $f\left(\frac{1}{4}\right) \approx f(0) + f'(0)\left(\frac{1}{4}\right)$

$= -1 + (2)\left(\frac{1}{4}\right) = -\frac{1}{2}$

$f\left(\frac{1}{2}\right) \approx f\left(\frac{1}{4}\right) + f'\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)$

$= -\frac{1}{2} + \left(-\frac{1}{2}\right)^2\left(2\cdot\frac{1}{4}+2\right)\left(\frac{1}{4}\right) = -\frac{11}{32}$

$2 : \begin{cases} 1 : \text{Euler's method} \\ 1 : \text{answer} \end{cases}$

(c) $\frac{dy}{dx} = y^2(2x+2)$

$\frac{dy}{y^2} = (2x+2)\,dx$

$\int\frac{dy}{y^2} = \int(2x+2)\,dx$

$-\frac{1}{y} = x^2 + 2x + C$

$-\frac{1}{-1} = 0^2 + 2\cdot 0 + C \Rightarrow C = 1$

$-\frac{1}{y} = x^2 + 2x + 1$

$y = -\frac{1}{x^2+2x+1} = -\frac{1}{(x+1)^2}$

$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables