Scoring Guidelines

2006 BC 3 (3.51)

(a) $a(2) = \langle 0.395 \text{ or } 0.396, -0.741 \text{ or } -0.740\rangle$ Speed $= \sqrt{x'(2)^2 + y'(2)^2} = 1.207$ or 1.208	$2: \begin{cases} 1: \text{acceleration} \\ 1: \text{speed} \end{cases}$
(b) $\sin^{-1}(1 - 2e^{-t}) = 0$ $1 - 2e^{-t} = 0$ $t = \ln 2 = 0.693$ and $\frac{dy}{dt} \neq 0$ when $t = \ln 2$	$2: \begin{cases} 1: x'(t) = 0 \\ 1: \text{answer} \end{cases}$
(c) $m(t) = \frac{4t}{1+t^3} \cdot \frac{1}{\sin^{-1}(1 - 2e^{-t})}$ $\lim_{t\to\infty} m(t) = \lim_{t\to\infty}\left(\frac{4t}{1+t^3} \cdot \frac{1}{\sin^{-1}(1 - 2e^{-t})}\right)$ $= 0\left(\frac{1}{\sin^{-1}(1)}\right) = 0$	$2: \begin{cases} 1: m(t) \\ 1: \text{limit value} \end{cases}$
(d) Since $\lim_{t\to\infty} x(t) = \infty$, $c = \lim_{t\to\infty} y(t) = -3 + \int_2^\infty \frac{4t}{1+t^3}\,dt$	$3: \begin{cases} 1: \text{integrand} \\ 1: \text{limits} \\ 1: \text{initial value consistent with lower limit} \end{cases}$

2007 BC3 (2.84)

(a) Area $= \frac{2}{3}\pi(2)^2 + \frac{1}{2}\int_{2\pi/3}^{4\pi/3}(3 + 2\cos\theta)^2\,d\theta$ $= 10.370$	$4: \begin{cases} 1: \text{area of circular sector} \\ 2: \text{integral for section of limaçon} \\ \quad 1: \text{integrand} \\ \quad 1: \text{limits and constant} \\ 1: \text{answer} \end{cases}$
(b) $\left.\frac{dr}{dt}\right\rvert_{\theta=\pi/3} = \left.\frac{dr}{d\theta}\right\rvert_{\theta=\pi/3} = -1.732$ The particle is moving closer to the origin, since $\frac{dr}{dt} < 0$ and $r > 0$ when $\theta = \frac{\pi}{3}$.	$2: \begin{cases} 1: \left.\frac{dr}{dt}\right\rvert_{\theta=\pi/3} \\ 1: \text{interpretation} \end{cases}$
(c) $y = r\sin\theta = (3 + 2\cos\theta)\sin\theta$ $\left.\frac{dy}{dt}\right\rvert_{\theta=\pi/3} = \left.\frac{dy}{d\theta}\right\rvert_{\theta=\pi/3} = 0.5$ The particle is moving away from the x-axis, since $\frac{dy}{dt} > 0$ and $y > 0$ when $\theta = \frac{\pi}{3}$.	$3: \begin{cases} 1: \text{expression for } y \text{ in terms of } \theta \\ 1: \left.\frac{dy}{dt}\right\rvert_{\theta=\pi/3} \\ 1: \text{interpretation} \end{cases}$

Solution	Scoring
(a) $r(0) = -1;\ r(\theta) = 0$ when $\theta = \frac{\pi}{3}$. Area of $S = \frac{1}{2}\int_0^{\pi/3} (1 - 2\cos\theta)^2\, d\theta$	$2 : \begin{cases} 1 : \text{limits and constant} \\ 1 : \text{integrand} \end{cases}$
(b) $x = r\cos\theta$ and $y = r\sin\theta$ $\frac{dr}{d\theta} = 2\sin\theta$ $\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta = 4\sin\theta\cos\theta - \sin\theta$ $\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta = 2\sin^2\theta + (1 - 2\cos\theta)\cos\theta$	$4 : \begin{cases} 1 : \text{uses } x = r\cos\theta \text{ and } y = r\sin\theta \\ 1 : \frac{dr}{d\theta} \\ 2 : \text{answer} \end{cases}$
(c) When $\theta = \frac{\pi}{2}$, we have $x = 0,\ y = 1$. $\left.\frac{dy}{dx}\right\|_{\theta=\frac{\pi}{2}} = \left.\frac{dy/d\theta}{dx/d\theta}\right\|_{\theta=\frac{\pi}{2}} = -2$ The tangent line is given by $y = 1 - 2x$.	$3 : \begin{cases} 1 : \text{values for } x \text{ and } y \\ 1 : \text{expression for } \frac{dy}{dx} \\ 1 : \text{tangent line equation} \end{cases}$

(a) Area $= \frac{1}{2}\int_{\pi/2}^{\pi} (r(\theta))^2 \, d\theta = 47.513$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits and constant} \\ 1 : \text{answer} \end{cases}$

(b) $-3 = r(\theta)\cos\theta = (3\theta + \sin\theta)\cos\theta$
$\theta = 2.01692$
$y = r(\theta)\sin(\theta) = 6.272$

$3 : \begin{cases} 1 : \text{equation} \\ 1 : \text{value of } \theta \\ 1 : y\text{-coordinate} \end{cases}$

(c) $y = r(\theta)\sin\theta = (3\theta + \sin\theta)\sin\theta$

$$\left.\frac{dy}{dt}\right|_{\theta=2\pi/3} = \left[\frac{dy}{d\theta}\cdot\frac{d\theta}{dt}\right]_{\theta=2\pi/3} = -2.819$$

The y-coordinate of the particle is decreasing at a rate of 2.819.

$3 : \begin{cases} 1 : \text{uses chain rule} \\ 1 : \text{answer} \\ 1 : \text{interpretation} \end{cases}$

2013 BC2 (3.44)

(a) Area $= 6\pi + \frac{1}{2}\int_{\pi/6}^{5\pi/6} (4 - 2\sin\theta)^2 \, d\theta = 24.709$ (or 24.708)	$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits and constant} \\ 1 : \text{answer} \end{cases}$
(b) $x = r\cos\theta \Rightarrow x(\theta) = (4 - 2\sin\theta)\cos\theta$ $x(t) = \left(4 - 2\sin\left(t^2\right)\right)\cos\left(t^2\right)$ $x(t) = -1$ when $t = 1.428$ (or 1.427)	$3 : \begin{cases} 1 : x(\theta) \text{ or } x(t) \\ 1 : x(\theta) = -1 \text{ or } x(t) = -1 \\ 1 : \text{answer} \end{cases}$
(c) $y = r\sin\theta \Rightarrow y(\theta) = (4 - 2\sin\theta)\sin\theta$ $y(t) = \left(4 - 2\sin\left(t^2\right)\right)\sin\left(t^2\right)$ Position vector $= \langle x(t), y(t) \rangle$ $= \left\langle \left(4 - 2\sin\left(t^2\right)\right)\cos\left(t^2\right), \left(4 - 2\sin\left(t^2\right)\right)\sin\left(t^2\right) \right\rangle$ $v(1.5) = \langle x'(1.5), y'(1.5) \rangle$ $= \langle -8.072, -1.673 \rangle$ (or $\langle -8.072, -1.672 \rangle$)	$3 : \begin{cases} 2 : \text{position vector} \\ 1 : \text{velocity vector} \end{cases}$

(a) $f(4) = \dfrac{1}{4^2 - 3\cdot 4} = \dfrac{1}{4}$ $\quad f'(4) = \dfrac{3 - 2\cdot 4}{\left(4^2 - 3\cdot 4\right)^2} = -\dfrac{5}{16}$ An equation for the line tangent to the graph of f at the point whose x-coordinate is 4 is $y = -\frac{5}{16}(x-4) + \frac{1}{4}$.	$2: \begin{cases} 1: \text{slope} \\ 1: \text{tangent line equation} \end{cases}$
(b) $f'(x) = \dfrac{4 - 2x}{\left(x^2 - 4x\right)^2}$ $\quad f'(2) = \dfrac{4 - 2\cdot 2}{\left(2^2 - 4\cdot 2\right)^2} = 0$ $f'(x)$ changes sign from positive to negative at $x = 2$. Therefore, f has a relative maximum at $x = 2$.	$2: \begin{cases} 1: \text{considers } f'(2) \\ 1: \text{answer with justification} \end{cases}$
(c) $f'(-5) = \dfrac{k - 2\cdot(-5)}{\left((-5)^2 - k\cdot(-5)\right)^2} = 0 \Rightarrow k = -10$	1 : answer
(d) $\dfrac{1}{x^2 - 6x} = \dfrac{1}{x(x-6)} = \dfrac{A}{x} + \dfrac{B}{x-6} \Rightarrow 1 = A(x-6) + Bx$ $x = 0 \Rightarrow 1 = A\cdot(-6) \Rightarrow A = -\dfrac{1}{6}$ $x = 6 \Rightarrow 1 = B\cdot(6) \Rightarrow B = \dfrac{1}{6}$ $\dfrac{1}{x(x-6)} = \dfrac{-1/6}{x} + \dfrac{1/6}{x-6}$ $\int f(x)\,dx = \int\left(\dfrac{-1/6}{x} + \dfrac{1/6}{x-6}\right)dx$ $= -\dfrac{1}{6}\ln\lvert x\rvert + \dfrac{1}{6}\ln\lvert x-6\rvert + C = \dfrac{1}{6}\ln\left\lvert\dfrac{x-6}{x}\right\rvert + C$	$4: \begin{cases} 2: \text{partial fraction decomposition} \\ 2: \text{general antiderivative} \end{cases}$